TRIGONOMETRIK AYNIYATLAR VA ULARNING ISBOTLARI

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Annotatsiya:*Ushbu mavzu o`quvchilar uchun sodda,ravon tilda bayon qilingan,ya`ni o`quvchi xech kimni yordamisiz o`zi o`qib tushuna oladi.Bundan tashqari mavzu uchun misollar ham,ularni yechish usullari ko`rsatib qo`yilgan.Bu mavzuni yosh o`qituvchilarga dars jarayonida qo`llashini maqsadga muvofiq deb o`ylayman va tavsiya beraman.*

**Tayanch so`z va iboralar:** *trigonometriya, ayniyat,formula,asosiy trigonometric ayniyatlar,qo`shish formulasi,keltirish formulalari ,ta`rif, tekislik,nuqta, aylana, radius, misollar,ta`rif,ifoda, tenglik,argument,burchak,yarim burchak, kasr, surat, maxraj, yig`indi, ayirma.*

TA’RIF:

Argumentning qabul qilishi mumkin bo’lgan barcha qiymatlarida to’g’ri bo’lgan trigonometrik tenglik **trigonometrik ayniyat** deyiladi.

I.ASOSIY TRIGONOMETRIK AYNIYATLARNI ESLATIB O’TAMIZ

1. $\cos^2\alpha + \sin^2\alpha = 1;$
2. $\text{ctg}\alpha = \dfrac{\cos\alpha}{\sin\alpha}$
3. $\text{tg}\alpha = \dfrac{\sin\alpha}{\cos\alpha}$
4. $tg\alpha ctg\alpha = 1$
5. $\sec\alpha = \dfrac{1}{\cos\alpha}$
6. $\text{co}\sec\alpha = \dfrac{1}{\sin\alpha}$
7. $1 + tg^2\alpha = \sec^2\alpha = \dfrac{1}{\cos^2\alpha}$
8. $1 + ctg^2\alpha = \cos ec^2\alpha = \dfrac{1}{\sin^2\alpha}$

Misollar. Quyidagi ayniyatlarni isbotlang.

1. $\dfrac{1+2\sin\alpha\cos\alpha}{\sin^2\alpha - \cos^2\alpha} = \dfrac{tg\alpha + 1}{tg\alpha - 1}$

Isbot $\dfrac{1+2\sin\alpha\cos\alpha}{\sin^2\alpha - \cos^2\alpha} = \dfrac{(\sin\alpha + \cos\alpha)^2}{(\sin\alpha - \cos\alpha)(\sin\alpha + \cos\alpha)} = \dfrac{\sin\alpha + \cos\alpha}{\sin\alpha - \cos\alpha}$

Kasrni surat va maxrajini $\cos\alpha \neq 0$ ga bo’lamiz, u holda

$$\frac{\sin\alpha+\cos\alpha}{\sin\alpha-\cos\alpha}=\frac{tg\alpha+1}{tg\alpha-1}$$

2. $$\sqrt{\frac{2}{1+\sin\alpha}+\frac{2}{1-\sin\alpha}}=\frac{2}{|\cos\alpha|}$$

Isbot:

$$\sqrt{\frac{2}{1+\sin\alpha}+\frac{2}{1-\sin\alpha}}=\sqrt{\frac{2(1-\sin\alpha)+2(1+\sin\alpha)}{1-\sin^2\alpha}}=\sqrt{\frac{4}{\cos^2\alpha}}=\frac{2}{|\cos\alpha|}$$

TRIGONOMETRIK AYNIYATLAR.

a) Qo'shish formulalari:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$tg(\alpha+\beta)=\frac{tg\alpha+tg\beta}{1-tg\alpha tg\beta}$$

$$tg(\alpha-\beta)=\frac{tg\alpha-tg\beta}{1+tg\alpha tg\beta},\qquad (\alpha+\beta\neq\frac{\pi}{2}+\pi k)$$

b) Tigonometrik funksiyalar yig'indisini va ayirmasini ko'paytmaga keltirish formulasi.

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$$

$$tg+tg\beta=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta};\qquad tg-tg\beta=\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$$

$$ctg+ctg\beta=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}\qquad;\quad ctg-ctg\beta=\frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$$

Trigonometrik ayniyatlarni isbotlashning quyidagi usullari mabjud.

1. Aynan shakl almashtirishlar yordamida tenglikning u yoki bu qismida turgan ifodani tenglikning ikkinchi qismdagi ifodaga keltiriladi.

2. Ayniyatning o'rta va chap qismidagi ifodalar bir xil ko'rinishga keltiriladi.

3. Aniyatning o'ng va chap qismida turgan ifodalar orasidagi ayirma nolga teng ekanligi ko'rsatiladi.

II. Endi ikki argument kosinuslarining ko'paytmasini yig'indiga keltirish formulalarini keltirib chiqaramiz.Ushbu ayniyatlarni hadlab qo'shamiz :

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\cos(\alpha+\beta)+\cos(\alpha+\beta)=2\cos\alpha\cos\beta$$

$$\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

Endi yuqoridagi ayniyatlarni hadlab ayiramiz.

$$\cos(\alpha-\beta)+\cos(\alpha+\beta)=2\sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

Ushbu ayniyatlarni ham qaraymiz.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

Bu ayniyatlarni hadlab qo'shamiz.:

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$$

$$\sin\alpha\sin\beta=\frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$$

a) Qo'shish formulalari .

sin($\alpha+\beta$) =sin $\alpha*\sin\beta$ +cos $\alpha*\cos\beta$;

sin($\alpha-\beta$)=sin $\alpha*\cos\beta$ -cos $\alpha*\sin\beta$;

cos($\alpha+\beta$)=cos $\alpha*\cos\beta$ -sin $\alpha*\sin\beta$;

cos($\alpha-\beta$)=cos $\alpha*\cos\beta$ +sin $\alpha*\sin\beta$;

tg($\alpha+\beta$ **)=** $\frac{tg\alpha+tg\beta}{1-tg\alpha tg\beta}$

tg($\alpha-\beta$ **)=** $\frac{tg\alpha-tg\beta}{1+tg\alpha tg\beta}$ **, (** $\alpha+\beta\neq\frac{\pi}{2}$ **+** $\pi\kappa$ **)**

b)Trigometrik funksiyalar yig'indisini va ayirmasini ko'paytmaga keltirish formulalari.

sin α +sin β =2sin $\frac{\alpha+\beta}{2}$ *cos $\frac{\alpha-\beta}{2}$

sin α **-sin** β **=2sin** $\frac{\alpha-\beta}{2}$ ***cos** $\frac{\alpha+\beta}{2}$

cos α **+cos** β **=2cos** $\frac{\alpha+\beta}{2}$ ***cos** $\frac{\alpha-\beta}{2}$

cos α **-cos** β **=-2sin** $\frac{\alpha+\beta}{2}$ ***sin** $\frac{\alpha-\beta}{2}$

tg α **+tg** β **=** $\frac{\sin(\alpha+\beta)}{\cos\alpha*\cos\beta}$ **; tg** $\alpha-tg\beta$ **=** $\frac{\sin(\alpha-\beta)}{\cos\alpha*\cos\beta}$ **;**

ctg $\alpha+ctg\beta$ **=** $\frac{\sin(\alpha+\beta)}{\sin\alpha*\sin\beta}$ **; ctg** $\alpha-ctg\beta$ **=** $\frac{\sin(\beta-\alpha)}{\sin\alpha*\sin\beta}$ **;**

tg $\alpha+ctg\beta$ **=** $\frac{\cos(\alpha-\beta)}{\cos\alpha*\sin\beta}$ **.**

4. a, b sonlar nolga teng emas.Tekislikda M(a,b) nuqtani olamiz. OM radius vektorni uzunligi :

$$\left|\overrightarrow{OM}\right| = R = \sqrt{a^2+b^2}: \quad u \ \ holda, \qquad \cos\varphi = \frac{a}{\sqrt{a^2+b^2}}; \qquad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}.$$

bunda, ф burchak OM ning absissalar o'qi bilan hosil qilgan burchagi

$$a\sin\alpha + b\cos\alpha = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin\alpha \pm \frac{b}{\sqrt{a^2+b^2}}\cos\alpha\right] = \sqrt{a^2+b^2}\sin(\alpha\pm\varphi)$$

Demak,

$$a\sin\alpha + b\cos\alpha = \sqrt{a^2+b^2}\sin(\alpha\pm\varphi)$$

Bunda, ф *yordamchi burchak* deyiladi.

N a m u n a. Ifodalarni almashtiring.

1. $\sin\alpha \pm \cos\alpha$;

Yechish.

$$\sin\alpha + \cos\alpha = \sqrt{2}\left(\sin\alpha\cdot\frac{\sqrt{2}}{2} + \cos\alpha\frac{\sqrt{2}}{2}\right) = \sqrt{2}\left(\sin\alpha\cdot\cos\frac{\pi}{4} + \cos\alpha\sin\frac{\pi}{4}\right) = \sqrt{2}\sin\left(\alpha+\frac{\pi}{4}\right)$$

2. $2\sin\alpha - 3\cos\alpha$

Yechish.

$$2\sin\alpha - \cos\alpha = \sqrt{13}\left(\frac{2}{\sqrt{13}}\sin\alpha\cdot - \frac{3}{\sqrt{13}}\cos\alpha\right) = \sqrt{13}\sin(\alpha-\varphi); \qquad \varphi = \arccos\frac{2}{\sqrt{13}}$$

3. $\sin\alpha - \cos\alpha$

Yechish:

$$\sin\alpha - \cos\alpha = \sqrt{2}\left(\sin\alpha\cdot\frac{1}{\sqrt{2}} - \cos\alpha\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\sin\alpha\cdot\cos\frac{\pi}{4} - \cos\alpha\sin\frac{\pi}{4}\right) = \sqrt{2}\sin\left(\alpha-\frac{\pi}{4}\right)$$

Demak,

$$\sin\alpha - \cos\alpha = \sqrt{2}\sin\left(\alpha-\frac{\pi}{4}\right) = \sqrt{2}\sin\left(\frac{\pi}{4}-\alpha\right)$$

4. $\sin x + \sqrt{3}\cos x$

Yechish.

$$\sin x + \sqrt{3}\cos x = \sqrt{1+3}\left(\frac{1}{\sqrt{1+3}}\sin x\cdot + \frac{\sqrt{3}}{\sqrt{1+3}}\cos x\right) = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right) = 2(\sin x\cos\frac{\pi}{4} +$$

$$+\sin\frac{\pi}{4}\cos x) = 2\sin(\frac{\pi}{3}+x)$$

Agar α=β bo'lsa, uholda

$\cos^2\alpha = \frac{1+\cos 2\alpha}{2};$

$\sin^2\alpha = \frac{1-\cos 2\alpha}{2};$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\sin\alpha\cos\alpha = \frac{\sin 2\alpha}{2}: \quad tg2\alpha = \frac{2tg\alpha}{1-tg^2\alpha}$

formulalar hosil bo’ladi.

5. Qo`shish formulasidan foydalanib, quyidagi ayniyatlarni isbotlang.

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

Ko’rsatma:

$\sin 3\alpha = \sin(2\alpha+\alpha)$ *va* $\cos 3\alpha = \cos(2\alpha+\alpha)$, *deb oling.*, $\sin 2\alpha = 2\sin\alpha\cos\alpha$ *va* $\cos 2\alpha =$
$= \cos^2\alpha - \sin^2\alpha$

Formulalardan foydalaning.

6. Yarim burchak trigonometrik funksiyalarlari formulalarini isbotlang.

$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}; \ \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}; \ tg\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$

Ko’rsatma: $\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha$ *va* $\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = 1$ ayniyatlardan foydalaning.

7. Ayniyatlarni isbotlang :

a) $\sin\alpha = \frac{tg\frac{\alpha}{2}}{1+tg^2\frac{\alpha}{2}};$

b) $\cos\alpha = \frac{1-tg^2\frac{\alpha}{2}}{1+tg^2\frac{\alpha}{2}};$

c) $tg\alpha = \frac{2tg\frac{\alpha}{2}}{1-tg^2\frac{\alpha}{2}};$

Ko’rsatma:

a) $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ *va* $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$ formulalardan foydalaning.

b) $\sin\alpha = \frac{\sin\alpha}{1}$ *va* $\cos\alpha = \frac{\cos\alpha}{1}$ *ko'rinishda yo'zing va* 1 *ni* $\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$ bilan almashtiring. Bunda $\alpha = \pi + 2\pi k$, *u holda* $\frac{\alpha}{2} \neq \frac{\pi}{2} + \pi k$, $k \in Z$. *Demak*, $\cos\frac{\alpha}{2} \neq 0$ ekanidan foydalanib, kasrning surat va maxrajini $\cos^2\frac{\alpha}{2}$ ga bo’lish mumkin.

3. Agar A+B+C=p bo'lsa, tgA+tgB+tgC=tgA·tgB·tgC ekanini ko'rsating.

Ko'rsatma:

$tg(A+B)=\dfrac{tgA+tgB}{1-tgAtgB}$ shartga ko'ra, tg(A+B)=tg(p-C)=-tgC, u holda :-

$$tgC=\frac{tgA+tgB}{1-tgAtgB}\Rightarrow -tgC+tgAtgBtgC=tgA+tgB+tgC\Rightarrow tgAtgBtgC$$

4. Agar A+B+C=π/2 bo'lsa, ctgA+ctgB+ctgC=ctgA·ctgB·ctgC ekanini ko'rsating.

Ko'rsatma:

$$ctg(A+B)=ctg(\frac{\pi}{2}-C)=tgC=\frac{1}{ctgC};\frac{1}{ctg\alpha}=\frac{ctgA\cdot ctgB-1}{ctgB+ctgA}\Rightarrow ctgB+ctgA=ctgActgBctgC-$$
$$ctgC\Rightarrow ctgA=ctgB+ctgC=ctgActgBctgC$$

Misol-1. Ayniyatlarni isbotlang.

a) $2\cos^2(\frac{\pi}{4}+\frac{\alpha}{2})=1-\sin\alpha$ b) $2\sin^2(\frac{\pi}{4}+\frac{\alpha}{2})=1+\sin\alpha$

d) $\dfrac{1-\cos\alpha}{\sin 2\alpha}\cdot ctg\alpha=1$ e) $\dfrac{\sin 2\alpha}{1+\cos 2\alpha}=tg\alpha$

Misol-2.

a) $\sin\alpha c\sin(\beta-\alpha)+\sin^2(\frac{\beta}{2}-\alpha)=\sin^2\frac{\beta}{2}$

b) $\cos^2\alpha-\sin^2 2\alpha=\cos^2\alpha+\cos 2\alpha\cdot 2\sin^2\alpha\cos^2\alpha$

Misol-3. Ifodani soddalashtiring.

a) $\dfrac{2(\cos\alpha+\cos 3\alpha)}{2\sin 2\alpha+\sin 4\alpha}$; b) $\dfrac{1+\sin\alpha-\cos 2\alpha-\sin 3\alpha}{2\sin 2\alpha+\sin\alpha-1}$

Misol-4. Ayniyatni isbotlang .

a) $tg+tg\beta=\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$; b) $tg267^0+tg93^0$

d) $tg\dfrac{5\pi}{12}+tg\dfrac{7\pi}{12}$