PUASSON TENGLAMASI UCHUN CHEGARADA YUQORI TARTIBLI HOSILALAR BERILGAN A MASALA

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Annotatsiya: *Ushbu maqolada Puasson tenglamasi uchun chegaraviy shartlarida yuqori tartibli hosilalar qatnashgan masala yechimining yagonaligi va mavjudligi o`zgaruvchilarni ajratish usuli yordamida isbotlangan.*

Kalit so`zlar: *elliptik tenglama, Puasson tenglamasi, masala yechimining yagonaligi, masala yechimining mavjudligi, Furye qatoriga yoyish.*

Masalani shakllantirish.

Elliptik tenglamalar uchun chegaraviy masalalar ko`plab tadqiqotchilar tomonidan keng o`rganilgan (masalan [1,2]). [3] ishda $(0<x<\infty, t>0)$ sohada issiqlik tarqalish tenglamasi uchun quyidagi

$$\sum_{k=1}^{m} a_k \frac{\partial^k u(0,t)}{\partial x^k} = f(x,t), \qquad u(x,0)=0$$

masala yechimining yagonaligi va mavjudligi isbotlangan. [4] ishda Laplas tenglamasi uchun n o`lchovli chegaralangan D sohada

$$\frac{d^m u}{dv^m} = f(x), \quad x \in \partial D$$

masala o`rganilgan va uni Fredgolmga tegishli ekanligi isbotlangan. Laplas, Puasson va Gelmgolts tenglamalari uchun birlik sharda chegaraviy shartlarda yuqori tartibli hosilalar berilgan masalalar Karachik [5-8], Sokolovskiy [9] va boshqalar tomonidan o`rganilgan. [4-8] ishlarda chegaraviy shartlar butun chegara bo`yicha berilgan. Shuning uchun masala yechimining yagonaligi ma`lum darajadagi bir jinsli ko`phadlar doirasida isbotlangan. Issiqlik tarqalish tenglamasi uchun to`g`ri to`rtburchak sohada boshlang`ich shartida yuqori tartibli hosila berilgan boshlang`ich-chegaraviy masala [10] ishda o`rganilgan.

$\Omega = \{(x,y): 0<x<p, 0<y<q\}$ sohada

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \tag{1}$$

tenglamani qaraymiz.

A masala. (1) tenglamaning $\frac{\partial^k}{\partial y^k} u(x,y) \in C(\overline{\Omega})$ sinfga tegishli va quyidagi

$$u(0,y) = \psi_1(y), \; 0 \le y \le q, \tag{2}$$

$$u\left(p,y\right)=\psi_2\left(y\right),\ 0\le y\le q, \qquad (3)$$

$$\frac{\partial^k u}{\partial y^k}\left(x,0\right)=\overline{\varphi}_{1k},\ 0\le x\le p, \qquad (4)$$

$$\frac{\partial^k u}{\partial y^k}\left(x,q\right)=\overline{\varphi}_{2k}\left(x\right),\ 0\le x\le p, \qquad (5)$$

shartlarni qanoatlantiruvchi $u\left(x,y\right)\in C_{x,y}^{2,2}\left(\overline{\Omega}\right)$ yechimi topilsin, bu yerda *k*-fiksirlangan (qat`iy) nomanfiy butun son, $\overline{\varphi}_{1k}\left(x\right)$, $\overline{\varphi}_{2k}\left(x\right)$, $\psi_1\left(y\right)$va $\psi_2\left(y\right)$ berilgan funksiyalar.

[18] ishda (2) va (3) chegaraviy shartlar bir jinsli bo`lgan hol o`rganilgan.

1-masala yechimining yagonaligi

1-teorema. Agar A masalaning yechimi mavjud bo`lsa u yagonadir.

Isbot. Faraz qilaylik $\varphi_{jk}\left(x\right)=0,\ \psi_j\left(y\right)=0,\quad j=\overline{1,2}\quad 0\le x\le p,\quad f\left(x,y\right)=0,$ $\left(x,y\right)\in\overline{\Omega}$ bo`lsin. $\overline{\Omega}$ da $u\left(x,y\right)=0$ ekanligini ko`rsatamiz.

$$\alpha_n\left(y\right)=\int_0^p u\left(x,y\right)X_n\left(x\right)dx \qquad (6)$$

integralni kiritamiz, bu yerda $X_n\left(x\right)=\sqrt{\frac{2}{p}}\sin\lambda_n x dx,\ \lambda_n=\frac{\pi n}{p},\ n=1,2,\ldots$ $L_2\left(0,p\right)$.

(6) ni *y* bo`yicha ikki marta differensiallab

$$\alpha_n''\left(y\right)=\int_0^p \frac{\partial^2 u}{\partial y^2}X_n\left(x\right)dx \qquad (7)$$

ni topamiz.

Bir jinsli tenglamadan quyidagi tenglik kelib chiqadi:

$$\alpha_n''\left(y\right)=-\int_0^p u_{xx}\left(x,y\right)X_n\left(x\right)dx$$

(7) ning o`ng tomonini (3) va (4) shartlarni hisobga olgan holda *y* bo`yicha ikki marta bo`laklab integrallab

$$\alpha_n''\left(y\right)-\lambda^2\alpha_n\left(y\right)=0$$

tenglamani hosil qilamiz.

Uning yechimi

$$\alpha_n\left(y\right)=C_{1n}e^{-\lambda_n y}+C_{2n}e^{\lambda_n y}$$

ko`rinishda yoziladi, bu yerda $C_{1n,}C_{2n}$ noma`lum doimiy koeffitsiyentlar. $C_{1n,}C_{2n}$ noma`lum koeffitsiyentlarni topishda quyidagi

$$\alpha_n^{(k)}\left(0\right)=0,\quad \alpha_n^{(k)}\left(q\right)=0 \qquad (8)$$

ga o`tadigan (4), (5) shartlardan foydalanamiz.

$\alpha_n^{(k)}(y)$ hosila

$$\alpha_n^{(k)}(x)=\lambda_n^k\left[(-1)^k C_{1n}e^{-\lambda_n y}+C_{2n}e^{\lambda_n y}\right]$$

ko`rinishga ega.

(8) dan foydalanib $C_{1n,}C_{2n}$ noma`lum koeffitsiyentlarni topishda quyidagi tenglamalar sistemasini hosil qilamiz:

$$\begin{cases}(-1)^k C_{1n}+C_{2n}=0\\(-1)^k C_{1n}e^{-\lambda_n q}+C_{2n}e^{\lambda_n q}=0.\end{cases}$$

Bu sistemaning determinanti $(-1)^k 2sh(\lambda_n q)\neq 0$ ga teng. Shuning uchun $C_{1n}=C_{2n}=0$. Bu yerdan $\alpha_n(y)=0$ ekanligi kelib chiqadi. $X_n(x)$ funksiyaning $L_2(0,p)$ da to`laligiga asosan $\overline{\Omega}$ da $u(x,y)=0$ ekanligi kelib chiqadi.

1-teorema isbotlandi.

A masala yechimining mavjudligi

(2)-(3) chegaraviy shartlar bir jinsli bo'lmaganligi sababli bu masalani yechish uchun, to'g'ridan to'g'ri o'zgaruvchilari ajratish usulini qo'llab bo'lmaydi. Lekin xususiy hosilali differensial tenglamalar nazariyasidan ma'lumki, bu masalani bir jinsli chegaraviy masalaga olib kelishimiz mumkin. Shuning uchun quyidagi yordamchi funksiyani kiritamiz:

$$w(x,y)=\frac{x}{p}\left[\psi_2(y)-\psi_1(y)\right]+\psi_1(y) \qquad (9)$$

Tekshirib ko'rish qiyin emaski

$$w(0,y)=\psi_1(y); \qquad w(p,y)=\psi_2(y),$$

Masalani yechimini yig'indi ko'rinishda yozamiz

$$u(x,y)=v(x,y)+w(x,y). \qquad (10)$$

Bu yerda $v(x,y)$ - yangi noma'lum funksiya. Shunday qilib (10) ga asosan biz keyingi masalaga keldik:

$\Omega=\{(x,y):0<x<p,0<y<q\}$, sohada

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=g(x,y) \qquad (11)$$

tenglamani qaraylik.

A` masala. (11) tenglamaning $\frac{\partial^k}{\partial y^k}v(x,y)\in C(\overline{\Omega})$ sinfga tegishli va quyidagi

$$v(0,y)=0,\ 0\le y\le q, \qquad (12)$$

$$v(p,y)=0,\ 0\le y\le q, \qquad (13)$$

$$\frac{\partial^k v}{\partial y^k}(x,0)=\varphi_{1k}(x),\ 0\le x\le p, \qquad (14)$$

$$\frac{\partial^k v}{\partial y^k}(x,q)=\varphi_{2k}(x),\ 0\le x\le p, \qquad (15)$$

shartlarni qanoatlantiruvchi yechimi topilsin, bu yerda

$$\frac{\partial^k}{\partial y^k}v(x,0)=\frac{\partial^k}{\partial y^k}u(x,0)-\frac{\partial^k}{\partial y^k}w(x,0)\equiv\varphi_{1k}(x), \qquad 0\le x\le p,$$

$$\frac{\partial^k}{\partial y^k}v(x,0)=\frac{\partial^k}{\partial y^k}u(x,0)-\frac{\partial^k}{\partial y^k}w(x,0)\equiv\varphi_{2k}(x), \qquad 0\le x\le p.$$

$$g(x,y)=f(x,y)-w_{xx}(x,y)-w_{yy}(x,y).$$

A\` masala yechimini

$$g(x,y)=\sum_{n=1}^{\infty}g_n(y)X_n(x). \qquad (16)$$

Furye qatori ko\`rinishda izlaymiz.

Ko\`rinib turibdiki $u(x,y)$ (2)-(3) shartlarni qanoatlantiradi. Faraz qilaylik

$$g(x,y)\in C^2(\Omega), \quad g(0,y)=g(p,y)=0, \quad \psi_1(x)\in C^2[0,p], \quad \psi_1(0)=\psi_1(p)=0$$

$$\psi_2(x)\in C^2[0,p], \quad \psi_2(0)=\psi_2(p)=0$$

bo\`lsin.

$g(x,y)$, $\varphi_{1k}(x)$, $\varphi_{2k}(x)$ funksiyalarni $X_n(x)$ funksiyalari bo\`yicha Furye qatoriga yoyamiz:

$$g(x,y)=\sum_{n=1}^{\infty}g_n(y)X_n(x), \qquad (17)$$

$$\varphi_{1k}(x)=\sum_{n=1}^{\infty}\varphi_{1n}X_n(x), \qquad (18)$$

$$\varphi_{2k}(x)=\sum_{n=1}^{\infty}\varphi_{2n}X_n(x), \qquad (19)$$

bu yerda

$$g_n(y)=\int_0^p g(x,y)X_n(x)dx, \qquad (20)$$

$$\varphi_{1n}=\int_0^p\varphi_{1k}(x)X_n(x)dx, \qquad (21)$$

$$\varphi_{2n}=\int_0^p\varphi_{2k}(x)X_n(x)dx. \qquad (22)$$

Ko\`rinib turibdiki $v(x,y)$ning yechimi (12), (13) shartlarni qanoatlantiradi

C_{1n}, C_{2n} koeffitsiyentlar quyidagi ko`rinishda topiladi:

$$C_{1n} = \frac{1}{\left(\sqrt{\lambda_n}\right)^k \left(e^{-\sqrt{\lambda_n}q} - e^{\sqrt{\lambda_n}q}\right)} \left[e^{-\sqrt{\lambda_n}q}\bar{\varphi}_{1n} - e^{-\sqrt{\lambda_n}q}\sum_{s=0}^{\frac{k-2}{2}} g_n^{(k-2-2s)}(0) - \right.$$

$$\left. -\bar{\varphi}_{2n} + \frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}(y-q)g_n(t)dt + \sum_{s=0}^{\frac{k-2}{2}} g_n^{k-2-2s}(q) \right] \qquad (23)$$

$$C_{2n} = \frac{1}{(-1)^k\left(\sqrt{\lambda_n}\right)^k \left(e^{\sqrt{\lambda_n}q} - e^{-\sqrt{\lambda_n}q}\right)} \left[e^{\sqrt{\lambda_n}q}\bar{\varphi}_{1n} - e^{\sqrt{\lambda_n}q}\sum_{s=0}^{\frac{k-2}{2}} g_n^{(k-2-2s)}(0) - \right.$$

$$\left. -\bar{\varphi}_{2n} + \frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}(y-q)g_n(t)dt + \sum_{s=0}^{\frac{k-2}{2}} g_n^{k-2-2s}(q) \right] \qquad (24)$$

Furye usulini qo`llab A` - masala yechimini k - toq bo`lganda

$$v(x,y) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{p}}\sin\frac{\pi n}{p}x\left\{ \frac{e^{\sqrt{\lambda_n}y}}{\left(\sqrt{\lambda_n}\right)^k\left(e^{-\sqrt{\lambda_n}q} - e^{\sqrt{\lambda_n}q}\right)}\left[e^{-\sqrt{\lambda_n}q}\left(\varphi_{1n} - \sum_{s=0}^{\frac{k-2}{2}} g_n^{(k-2-2s)}(0)\right) - \right.\right.$$

$$\left. -\varphi_{2n} + \frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}(y-t)g_n(t)dt + \sum_{s=0}^{\frac{k-2}{2}} g_n^{(k-2-2s)}(q) \right] +$$

$$+\frac{e^{-\sqrt{\lambda_n}y}}{(-1)^k\left(\sqrt{\lambda_n}\right)^k\left(e^{\sqrt{\lambda_n}q} - e^{-\sqrt{\lambda_n}q}\right)}\left[e^{\sqrt{\lambda_n}q}\left(\varphi_{1n} - \sum_{s=0}^{\frac{k-2}{2}} g_n^{(k-2-2s)}(0)\right) - \varphi_{2n} + \right.$$

$$\left. +\frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}(y-t)g_n(t)dt + \sum_{s=0}^{\frac{k-2}{2}} g_n^{(k-2-2s)}(q) \right] + \frac{1}{\sqrt{\lambda_n}}\int_0^y sh\sqrt{\lambda_n}(y-t)g_n(t)dt \quad (25)$$

ko`rinishda va $k$ - juft bo`lsa

$$v(x,y) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{p}}\sin\frac{\pi n}{p}x\left\{ \frac{e^{\sqrt{\lambda_n}y}}{\left(\sqrt{\lambda_n}\right)^k\left(e^{-\sqrt{\lambda_n}q} - e^{\sqrt{\lambda_n}q}\right)}\left[e^{-\sqrt{\lambda_n}q}\left(\varphi_{1n} - \sum_{s=0}^{\frac{k-2}{2}} g_n^{(k-2-2s)}(0)\right) - \right.\right.$$

$$-\varphi_{2n}+\frac{1}{\sqrt{\lambda_n}}\int_0^y sh\sqrt{\lambda_n}\left(y-t\right)g_n\left(t\right)dt+\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(q\right)\Bigg]+$$

$$+\frac{e^{-\sqrt{\lambda_n}y}}{\left(-1\right)^k\left(\sqrt{\lambda_n}\right)^k\left(e^{\sqrt{\lambda_n}q}-e^{-\sqrt{\lambda_n}q}\right)}\Bigg[e^{\sqrt{\lambda_n}q}\left(\varphi_{1n}-\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(0\right)\right)-\varphi_{2n}+$$

$$+\frac{1}{\sqrt{\lambda_n}}\int_0^y sh\sqrt{\lambda_n}\left(y-t\right)g_n\left(t\right)dt+\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(q\right)\Bigg]+\frac{1}{\sqrt{\lambda_n}}\int_0^y sh\sqrt{\lambda_n}\left(y-t\right)g_n\left(t\right)dt\Bigg\}$$

(26)

ko\`rinishda topamiz.

A-masala yechimini (3.10) yig\`indi ko\`rinishida topamiz. Agar k-toq bo\`lsa yechimni

$$u\left(x,y\right)=\sum_{n=1}^{\infty}\sqrt{\frac{2}{p}}\sin\frac{\pi n}{p}x\Bigg\{\frac{e^{\sqrt{\lambda_n}y}}{\left(\sqrt{\lambda_n}\right)^k\left(e^{-\sqrt{\lambda_n}q}-e^{\sqrt{\lambda_n}q}\right)}\Bigg[e^{-\sqrt{\lambda_n}q}\left(\varphi_{1n}-\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(0\right)\right)-$$

$$-\varphi_{2n}+\frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}\left(y-t\right)g_n\left(t\right)dt+\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(q\right)\Bigg]+$$

$$+\frac{e^{-\sqrt{\lambda_n}y}}{\left(-1\right)^k\left(\sqrt{\lambda_n}\right)^k\left(e^{\sqrt{\lambda_n}q}-e^{-\sqrt{\lambda_n}q}\right)}\Bigg[e^{\sqrt{\lambda_n}q}\left(\varphi_{1n}-\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(0\right)\right)-\varphi_{2n}+$$

$$+\frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}\left(y-t\right)g_n\left(t\right)dt+\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(q\right)\Bigg]+$$

$$+\frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}\left(y-t\right)g_n\left(t\right)dt+\frac{x}{p}\left[\psi_2\left(y\right)-\psi_1\left(y\right)\right]+\psi_1\left(y\right) \qquad (27)$$

ko\`rinishda topamiz, agar $k$ –juft bo\`lsa yechimni

$$u\left(x,y\right)=\sum_{n=1}^{\infty}\sqrt{\frac{2}{p}}\sin\frac{\pi n}{p}x\Bigg\{\frac{e^{\sqrt{\lambda_n}y}}{\left(\sqrt{\lambda_n}\right)^k\left(e^{-\sqrt{\lambda_n}q}-e^{\sqrt{\lambda_n}q}\right)}\Bigg[e^{-\sqrt{\lambda_n}q}\left(\varphi_{1n}-\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(0\right)\right)-$$

$$-\varphi_{2n}+\frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}\left(y-t\right)g_n\left(t\right)dt+\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}\left(q\right)\Bigg]+$$

$$+\frac{e^{-\sqrt{\lambda_n}y}}{(-1)^k\left(\sqrt{\lambda_n}\right)^k\left(e^{\sqrt{\lambda_n}q}-e^{-\sqrt{\lambda_n}q}\right)}\left[e^{\sqrt{\lambda_n}q}\left(\varphi_{1n}-\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}(0)\right)-\varphi_{2n}+\right.$$

$$\left.+\frac{1}{\sqrt{\lambda_n}}\int_0^y ch\sqrt{\lambda_n}(y-t)g_n(t)dt+\sum_{s=0}^{\frac{k-2}{2}}g_n^{(k-2-2s)}(q)\right]+$$

$$+\frac{1}{\sqrt{\lambda_n}}\int_0^y sh\sqrt{\lambda_n}(y-t)g_n(t)dt+\frac{x}{p}\left[\psi_2(y)-\psi_1(y)\right]+\psi_1(y) \qquad (28)$$

ko`rinishda topamiz.

ADABIYOTLAR:

1. Mizohata, S: The Theory of Partial Differential Equations. Cambridge University Press, New York (1973)

2. Г.Н.Положий. Уравнения математической физики. Издательство Высшая школа, Москва, 1964

3. А.Н.Тихонов: О граничных условиях, содержащих производные порядка, превышающего порядок уравнения. Мат.сб. 26(1), 35-56 (1950)

4. Бицадзе А.В. К задаче Неймана для гармонических функций. Докл. АН СССР, 1990.-N.311.-№1. Стр. 11-13

5. Карачик В.В., Турметов Б.К. Задача для гармонического уравнения. Изв. Акад. Наук УзССР. Сер. Физ.-Мат. Наук 4, 17-21, 89 (1990)

6. В.В.Карачик. О разрешимости краевой задачи для уравнения Гельмгольца с нормальными производными высокого порядка на границе. Диффер. Уравн. **28**(5), 907-909, 920 (1992)

7. Karachik, VV: On a problem for the Poisson equation with high-order normal derivative on the boundary. Differ.Uravn. **32**(3), 416-418, 431 (1996) (in Russian); translation in Differ. Equ. **32**(3), 421-424 (1996)

8. В.В.Карачик. Обобщенная задача Неймана для гармонических функций в полупространстве. Диффер.Уравн. 35(7), 942-947 (1999); translation in Differ. Equ. **35**(7), 949-955 (1999)

9. В.Б.Соколовский. Об одном обобщении задачи Неймана. Диффер. Уравн. 24(4), 714-716, (1988)

10. Amanov, D: On a generalization of the first initial-boundary value problem for the heat conduction equation. Contemp. Anal. Appl. Math. **2**(1), 88-97 (2014)

11. Бердышев А.С., Азизов М.С. Смешанная задача для уравнения четвертого порядка с сингулярным коэффициентом в прямоугольнике // Научный вестник ФерГУ. 2019. №2. – С. 10-19.

12. Уринов А.К., Азизов М.С. Краевая задача для уравнения четвертого порядка с сингулярным коэффициентом в прямоугольнике // Научный вестник НамГУ. 2019. №11. – С. 26-37.

13. Azizov M.S. A boundary problem for the fourth order equation with a singular coefficient in a rectangular region // Lobachevskii Journal of Mathematics, Vol. 41, №6. 2020. pp. 1043-1050.

14. Азизов М.С. Смешанная задача для неоднородного уравнения четвертого порядка с сингулярным коэффициентом в прямоугольнике // Бюллетень Института математики 2020. №4, -С. 50-59.

15. Urinov A.K., Azizov M.S. Boundary problem for the loaded partial differential equations of fourth order // Lobachevskii Journal of Mathematics, 2021 Vol. 42, №3. pp. 621-631.

16. Urinov A.K., Azizov M.S. Boundary value problems for a fourth order partial differential equation with an unknown right-hand part // Lobachevskii Journal of Mathematics, 2021 Vol. 42, №3. pp. 632-640.

17. Азизов М.С. Обратная задача для уравнения четвертого порядка с сингулярным коэффициентом // Бюллетень Института математики 2021. Т.4., №4, -С. 51-60.

18. Amanov D. On a generalization of the Dirichlet problem for the Poisson equation // Boundary Value Problems, 2016 № 2016:160, pp. 170-182.

19. Е.И.Моисеев. О решении спектральным методом одной нелокальной краевой задачи. Диффер. Уравн. 35(8), 1094-1100, (1999)

20. Ильин В.А., Позняк Э.Г. Основы математического анализа. Часть 2. Наука, Москва, 1973

21. Бари Н.К. Тригонометрические ряды. Изд. Физ.-Мат.Лит., Москва 1961

22. Люстерник Л.А., Соболев В.И. Элементы функционального анализа. Наука.Москва.1965.